

Do Solar system tests permit higher dimensional general relativity?

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ABSTRACT

We perform a survey whether higher dimensional Schwarzschild space-time is compatible with some of the solar system phenomena. As a test we examine five well known solar system effects, viz., (1) Perihelion shift, (2) Bending of light, (3) Gravitational redshift, (4) Gravitational time delay and (5) Motion of test particle in the framework of general relativity with higher dimensions. It is shown that the results related to all these physical phenomena are mostly incompatible with the higher dimensional version of general relativity except that of Motion of test particle. We compare all these results with the available data in the literature.

Key words: gravitation - Solar system: general - celestial mechanics.

1 INTRODUCTION

Einstein's general relativity (GR) received first instant success due to the observational confirmation of two solar system effects, firstly, contribution to perihelion shift of 43 arcsec per century as curvature effect of space-time, and secondly, total solar eclipse in the year 1919 which admits the relativistic value 1.75 arcsec as obtained by Einstein which is also due to the effect of curvature. Probably these observational boldness and the sublime structure of GR inspired Born (1962) to state that, "*The theory appeared to me then, and it still does, the greatest feat of human thinking about nature, the most amazing combination of philosophical penetration, physical intuition, and mathematical skill. It appealed to me like a great work of art ...*".

The above mentioned two triumph of GR is obviously based on its usual four-dimensional structure of space-time. This prompted people to start thinking of the multidimensional structure of GR. However, the extension of GR by the inclusion of dimensions beyond four were initiated by investigators mainly in connection to the studies of early Universe. It is commonly believed that the four-dimensional present space-time is the compactified form of manifold with higher dimensions (HD). This self-compactification of multidimensions have been thought of by several researchers (Schwarz 1985; Weinberg 1986) in the area of grand unification theory as well as in superstring theory.

In the Kaluza-Klein gravitational theory with higher dimensions, therefore, it is a common practice to show that extra dimensions are reducible to lower one, specially in four-dimension which was associated with some physical processes. Interestingly, mass have been considered as the fifth dimension (Wesson 1983;

Fukui 1987; Banerjee, Bhui & Chatterjee 1990; Chatterjee & Bhui 1990; Ponce de Leon 2003) in the case of five-dimensional Kaluza-Klein theory. Fukui (1987) suggested that expansion of the Universe follows by the percolation of radiation into 4-dimensional space-time from the fifth dimensional mass. Ponce de Leon (2003) argued that the rest mass of a particle, perceived by an observer in four-dimension, varies as a result of the five-dimensional motion along the extra direction and in the presence of electromagnetic field is totally of electromagnetic origin which has confirmed by Ray (2006). On the other hand, it have been shown by many investigators (Ishihara 1984; Gegenberg & Das 1985) that within the Kaluza-Klein inflationary scenario of HD a contraction of the internal space causes the inflation of the usual space. There are cases in FRW cosmologies where the extra dimensions contract as a result of cosmological evolution (Ibañez & Verdaguer 1986). In the solution to the vacuum field equations of GR in $4 + 1$ dimensions Chodos & Detweiler (1980) have shown that it leads to a cosmology which at the present epoch has $3 + 1$ observable dimensions in which the Einstein-Maxwell equations are obeyed.

Under these theoretical background, therefore, now-a-days people have started to think of the higher dimensional influence on GR, more precisely, whether within the framework of higher dimensional GR the same type of solar system tests would yield the same results. Actually, it has two-fold intentions: firstly, if the results are positive then the higher dimensional version of GR will prove itself as an extended viable theory of gravitation, and secondly, if negative then there is no need of higher dimensional GR at all. Motivated by this, therefore, in a recent work Liu & Overduin (2000) argued that to test the theory involving the motion of test particles in the field of a static spherically-symmetric mass like the Sun or the Earth would be most straightforward. Kagramanova, Kunz & Lämmerzahl (2006) have investigated Solar system effects in Schwarzschild-Sitter space-time and estimated

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the values for the cosmological parameter Λ . In a similar line of thinking Iorio (2005a,b) attempted to investigate secular increase of the Astronomical Unit, perihelion precessions and planetary motions as tests of the Dvali-Gabadadze-Porrati multidimensional braneworld scenario.

In this connection it is to be noted here that investigations by Liu & Overduin (2000), along with those of Lim, Overduin & Wesson (1995) and Kalligas, Wesson & Everitt (1995), are limited to five-dimensional soliton-like space-time only. Therefore, our present attempt is to study more general cases under a spherically symmetric Schwarzschild-like space-time with N number of dimensions where $N = D + 2$ such that $D \geq 2$. In this context we discuss the following five cases involved in the solar system experiments to examine the viability of GR with HD, viz., (1) Perihelion shift (2) Bending of light (3) Gravitational Red-shift (4) Gravitational time delay and (5) Motion of test particle. Our present studies show that most of these solar system phenomena do not allow dimensions beyond 4 indicating a gross failure of GR with higher dimensional framework.

2 MATHEMATICAL FORMULATION

Let us consider a spherically symmetric metric which represents a generalized Schwarzschild space-time with higher dimensions (Mayers & Perry 1986)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_D^2, \quad (1)$$

where r is a radial coordinate and f is a function of r only. The line element $d\Omega_D^2$ on the unit D -sphere is given by

$$d\Omega_D^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2 + \dots + \prod_{n=1}^{D-1} \sin^2\theta_n d\theta_D^2 \quad (2)$$

with $\Omega_D = 2[\pi^{(D+1)/2}]/[\Gamma(D+1)/2]$. Also, according to Einstein equations we can write $f(r) = 1 - \mu/r^{D-1}$ with the constant of integration $\mu = 16\pi GM/De^2\Omega_D$.

Now, in principle, in Lagrangian mechanics the trajectory of an object is derived by finding the path which minimizes the action, a quantity which is the integral of the Lagrangian over time. So, in connection to the solar system problem we would like to adopt the higher dimensional Lagrangian which can be written as

$$L = T - V = -f\dot{t}^2 + \frac{\dot{r}^2}{f} + r^2\dot{\theta}_1^2 + r^2\sin^2\theta_1\dot{\theta}_2^2 + \dots + r^2 \prod_{n=1}^{D-1} \sin^2\theta_n \dot{\theta}_D^2. \quad (3)$$

Here dot over any parameter implies differentiation with respect to the affine parameter 's'.

Now, if we take a cross-section by keeping fixed $\theta_1 = \theta_2 = \dots = \theta_{D-1} = \frac{\pi}{2}$, so that $\dot{\theta}_i = 0, i = 1, 2, 3, \dots, D-1$ then the Lagrangian takes the form

$$L = -f\dot{t}^2 + \frac{\dot{r}^2}{f} + r^2\dot{\theta}_D^2 \quad (4)$$

with light-like particle photon, $L = 0$ and for any time-like particle, $L = 1$.

Therefore, in terms of the generalized coordinates q_i and generalized velocities \dot{q}_i , the standard Euler-Lagrange equations are

$$\frac{d}{ds} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0. \quad (5)$$

By assuming $f\dot{t} = E$ and $r^2\dot{\theta}_D = p$, where E and p are the energy and momentum of the particle respectively, such that $\dot{t} = E/f$ and $\dot{\theta}_D = p/r^2$ and hence with these notations equation (4) becomes

$$L = -\frac{E^2}{f} + \frac{\dot{r}^2}{f} + \frac{p^2}{r^2} \quad (6)$$

which, after simplification, can be written in the following forms

$$\dot{r}^2 = Lf + E^2 - \frac{p^2 f}{r^2} \quad (7)$$

and

$$\frac{1}{r^4} \left(\frac{dr}{d\theta_D} \right)^2 = \frac{Lf}{p^2} + \frac{E^2}{p^2} - \frac{f}{r^2}. \quad (8)$$

Again, by substituting $\theta_D = \phi$ and $r = 1/U$ in equation (8), one can write

$$\left(\frac{dU}{d\phi} \right)^2 = \frac{Lf}{p^2} + \frac{E^2}{p^2} - fU^2. \quad (9)$$

Now, if we write equation (7) in the form

$$\left(\frac{dr}{dt} \right)^2 = \frac{Lf^3}{E^2} + f^2 - \frac{p^2 f^3}{E^2 r^2} \quad (10)$$

then one can easily observe that $\frac{dr}{dt}$ vanishes at $r = r_0$ of the closest approach to the sun. This at once yields the relationship between momentum and energy of the particle as follows: $p^2/E^2 = r_0^2/f_0$, where $f_0 = f(r = r_0)$. Hence, the equation of photon becomes

$$\left(\frac{dr}{dt} \right)^2 = f^2 - \frac{f^3 r_0^2}{f_0 r^2}. \quad (11)$$

Thus, the time required for light to travel from r_0 to r can be expressed as

$$t(r, r_0) = \int_{r_0}^r \frac{dr}{\left[f^2 - \frac{f^3 r_0^2}{f_0 r^2} \right]^{1/2}}. \quad (12)$$

3 SOLAR SYSTEM TESTS FOR HIGHER DIMENSIONAL GR

3.1 Perihelion shift

Following the equation (8), motion of planet in the sun's gravitational field can be written as

$$\frac{1}{r^4} \left(\frac{dr}{d\phi} \right)^2 = \frac{f}{p^2} + \frac{E^2}{p^2} - \frac{f}{r^2}. \quad (13)$$

For $r = 1/U$, we have

$$\frac{d^2 U}{d\phi^2} + U = \mu(D+1)U^D + \frac{\mu}{p^2}(D-1)U^{D-2}. \quad (14)$$

The solution to this equation (14) is then given by the following cases:

(i): $D = 2$

By the use of successive approximation (taking $\mu = 0$ as zeroth approximation) we get the solution to the above equation (14) in the form

$$U = \frac{1}{l}(1 + e \cos \phi) \quad (15)$$

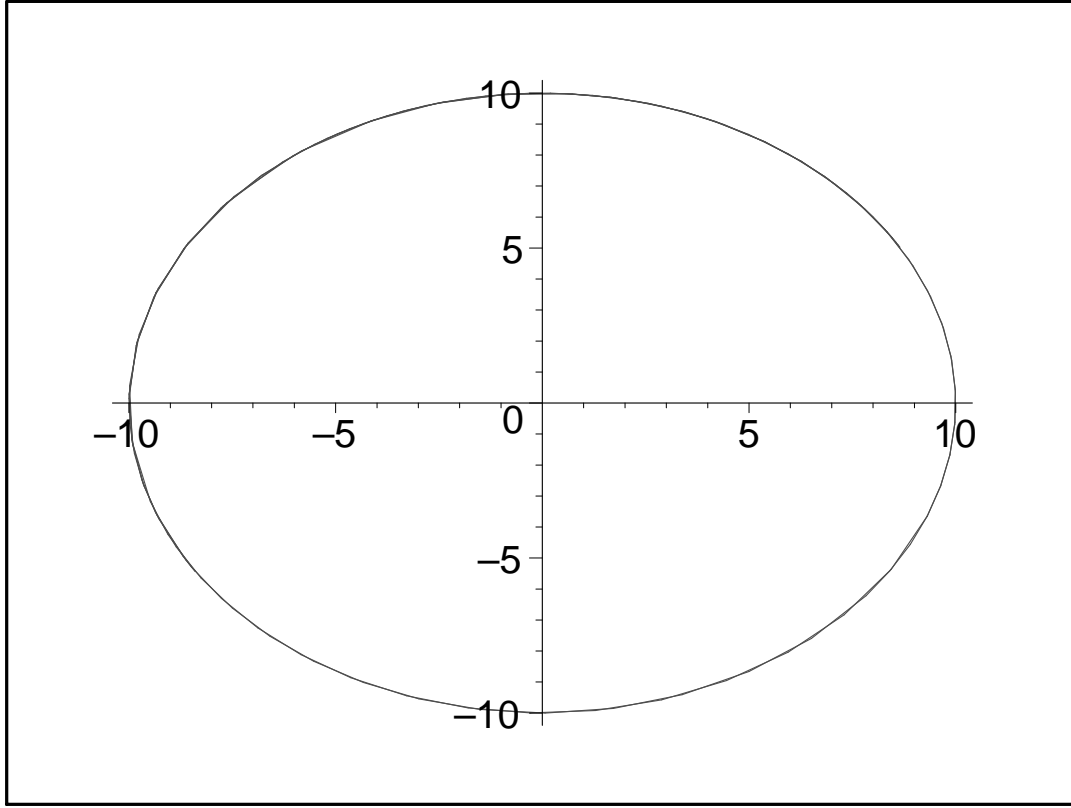


Figure 1. The trajectory of equation (15) by choosing suitably the parameters.

where $l = p^2 / \frac{GM}{c^2}$. Obviously, the trajectory of test particle, i.e., planet is elliptical (see Fig. 1).

Substituting this on the right hand side for U , we get

$$U = \frac{1}{l} [1 + e \cos(\phi - \omega)] \quad (16)$$

with $\omega = (3GM/c^2 l) \phi$. Therefore, time period for the planet is $T = 2\pi(3GM/c^2 l)$ and the average precession can be obtained as $n = 6\pi GM/c^2 l T = 43.03$ arcsec per century (where $l = 5.53 \times 10^{12}$ cm, $GM/c^2 = 1.475 \times 10^5$ cm and one century = $415T$). This value of Mercury's perihelion precession rate is very close to some of the available data which are 43.11 ± 0.21 and 42.98 arcsec per century respectively as obtained by Shapiro, Counselman & King (1976) and Liu & Overduin (2000).

(ii): $D = 3$

The equation of motion in this case can be written as

$$\frac{d^2 U}{d\phi^2} + U(1 - \frac{2\mu}{p^2}) = 4\mu U^3. \quad (17)$$

The solution is given by

$$U = U_0 \cos(\beta\psi) + U_1 \cos(3\beta\psi) \quad (18)$$

where $\psi = a\phi$, $U_1 = -(2\mu/16a)U_0^2 \ll U_0$ and $\beta^2 = 1 - (3\mu/a)U_0^2$ with $a = 1 - 2\mu/p^2$. Hence, the path is no longer elliptical (see Fig. 2).

(iii): $D = 4$

The solution of the equation (14) related to motion of planet in this case is

$$U = \frac{\cos\phi}{r_0} + \frac{5\mu}{r_0^4} \left[\frac{3}{8} - \frac{1}{6}(2\cos^2\phi - 1) - \frac{1}{120}(8\cos^4\phi - 8\cos^2\phi - 1) + \frac{3\mu}{p^2 r_0^2} \left[\frac{1}{2} - \frac{1}{3}(2\cos^2\phi - 1) \right] \right]. \quad (19)$$

Again, one can observe that the path is no longer elliptical (see Fig. 3).

Einstein (1915) explained the perihelion motion of mercury from the general theory of relativity by accounting the unsolved amount of ~ 43 arcsec as due to gravitation being mediated by the curvature of spacetime (Nordvedt 2001). Besides this relativistic effect other effects due to classical reasons are shown in the Table 1. Therefore, from the present investigation it is revealed that the HD model of general relativity only admit four-dimensional case with a precession 43.03 arcsec per century.

3.2 Bending of Light

Now we would like to observe how higher dimensional version of general relativity do respond on the effect of light bending. Let us, therefore, start with the equation (9) which now reads

$$\left(\frac{dU}{d\phi} \right)^2 = \frac{E^2}{p^2} - U^2(1 - \mu U^{D-1}). \quad (20)$$

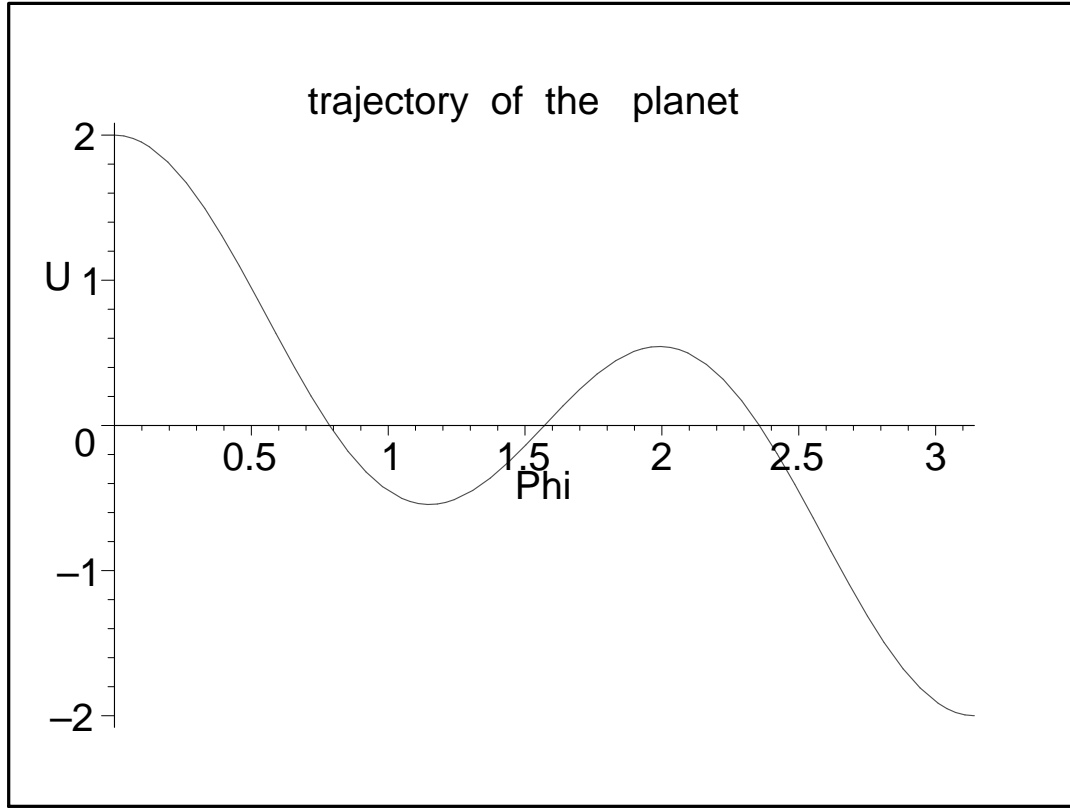


Figure 2. The plot U vs. ϕ for $D = 3$.

Table 1. Sources of the perihelion shift

Amount	Cause (arcsec per century)
5025.6	Precession of equinoxes
531.4	Gravitational tugs of the other planets
0.0254	Oblateness of the Sun
42.98 ± 0.04	Relativistic curvature of spacetime
5600.0	Total
5599.7	Observed

The above equation can be written in the suitable form as

$$\frac{d^2 U}{d\phi^2} + U = \frac{\mu(D+1)}{2} U^D. \quad (21)$$

Now, we solve the equation by successive approximation, starting with the straight line (path without gravitating body) as zeroth approximation such that $U = \cos\phi/R_0$ where $\phi = 0$ is the point P of nearest approach to the Sun's surface. Ideally, R_0 would be the solar radius.

Substituting this on the right hand side of equation (21) for U , we get

$$\frac{d^2 U}{d\phi^2} + U = \frac{\mu(D+1)}{2R_0^D} \cos^D \phi. \quad (22)$$

The solution of the above equation (22) is then given by for the following cases:

Case I: $D = \text{even} = 2n$

Let us consider the case when D is even and takes the value $2n$. For this particular situation the solution to the equation (22) can be given as

$$U = \frac{\cos\phi}{R_0} + \frac{\mu(2n+1)}{2^{2n}R_0^{2n}} \left[\frac{\cos 2n\phi}{-4n^2+1} + \frac{{}^{2n}C_1 \cos(2n-2)\phi}{-(2n-2)^2+1} + \dots + \frac{{}^{2n}C_{n-1} \cos 2\phi}{-2^2+1} \right] + \frac{\mu(2n+1){}^{2n}C_n}{2^{2n+1}R_0^{2n}}. \quad (23)$$

(i): For $n = 1$

In this subcase

$$U = \frac{\cos\phi}{R_0} + \frac{GM}{R_0^2 c^2} (2 - \cos^2 \phi). \quad (24)$$

For $U = 0$, we get $\cos\phi = -0.4244302380 \times 10^{-5}$ where the values for the constants are taken as follows: $c = 2.997925 \times 10^8$ m/sec, $G = 6.67323 \times 10^{-11}$ SI Unit, $M = 1.9892 \times 10^{30}$ Kg and $R_0 = 6.95987 \times 10^8$ m. We plot U vs. ϕ (see Fig. 4).

Here the net deflection of the ray is given by

$$\Delta\phi = 1.741300716 \text{ arcsec}. \quad (25)$$

This result is in agreement with the experimental result of observed deflection of light by the Sun (see Table 2).

(ii): $n = 2$

Here

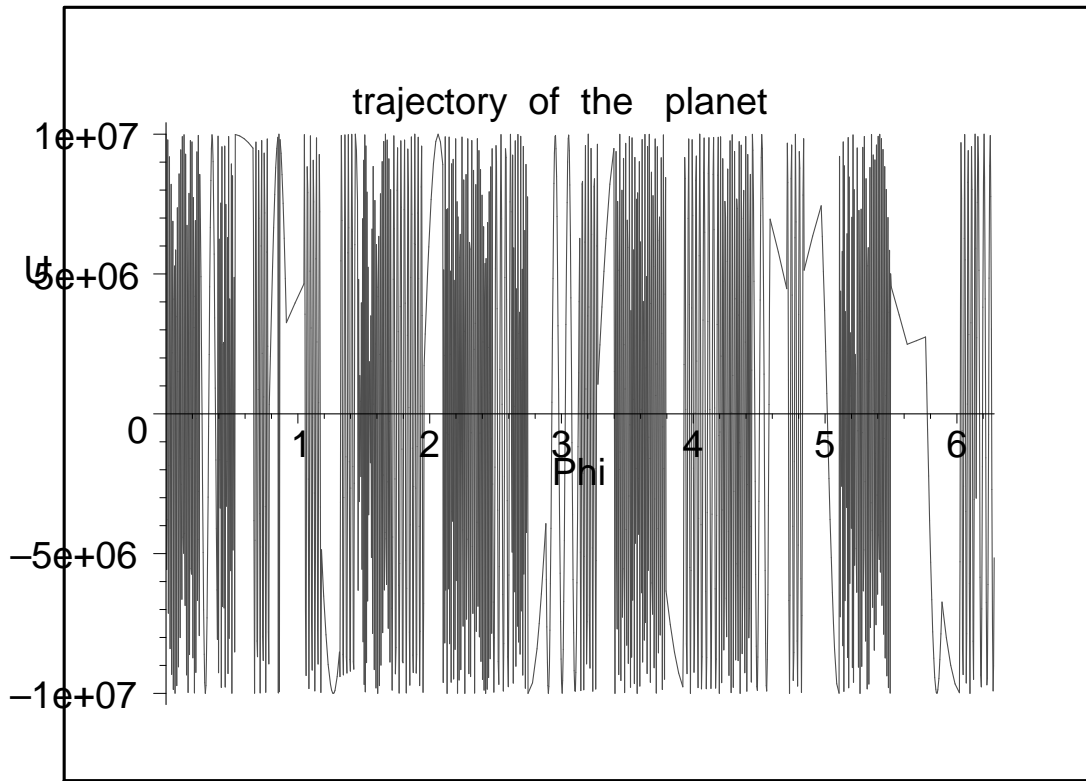


Figure 3. The plot U vs. ϕ for $D = 4$.

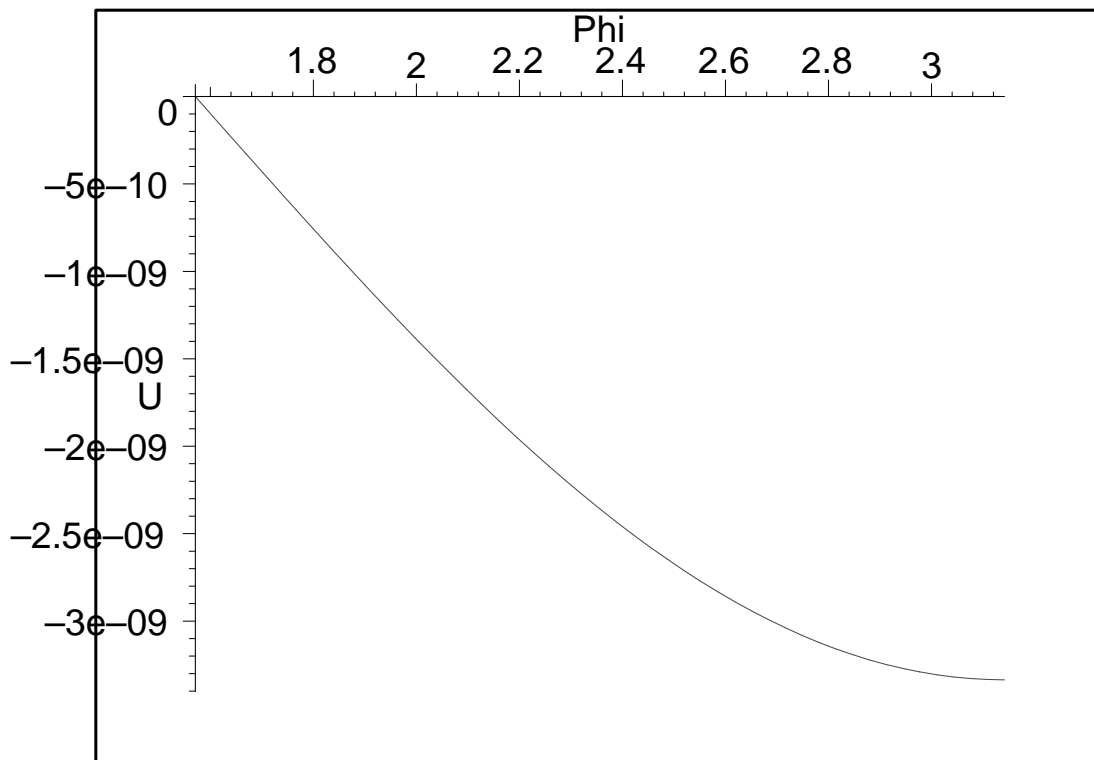


Figure 4. The plot U vs. ϕ for $D = 2$.

Table 2. Deflection of Starlight During Eclipses

Date	Location	Deflection ($\Delta\phi$) (arcsec)
29 May 1919	Sobral	1.98 ± 0.16
	Principe	1.61 ± 0.40
21 Sep 1922	Australia	1.77 ± 0.40
		$1.42 \text{ to } 2.16$
		1.72 ± 0.15
		1.82 ± 0.20
9 May 1929	Sumatra	2.24 ± 0.10
19 June 1936	USSR	2.73 ± 0.31
	Japan	$1.28 \text{ to } 2.13$
20 May 1947	Brazil	2.01 ± 0.27
25 Feb 1952	Sudan	1.70 ± 0.10
30 June 1973	Mauritania	1.66 ± 0.19

$$U = \frac{\cos\phi}{R_0} - \frac{\mu}{6R_0^4}\cos^4\phi - \frac{2\mu}{3R_0^4}\cos^2\phi + \frac{4\mu}{3R_0^4}. \quad (26)$$

For $U = 0$, we get $\cos\phi = \text{imaginary}$. We plot U vs. ϕ (see Fig. 5).

(iii): $n = 3$

For the value $n = 3$, we get

$$U = \frac{\cos\phi}{R_0} - \frac{7\mu}{64R_0^6} \left[\frac{1}{35}(32\cos^6\phi - 48\cos^4\phi + 8\cos^2\phi - 1) \right. \\ \left. + \frac{2}{5}(8\cos^4\phi - 8\cos^2\phi + 1) + 5(2\cos^2\phi - 1) \right] + \frac{35\mu}{32R_0^6} \quad (27)$$

Here $U = 0$ yields $\cos\phi = -0.2255226204 \times 10^{-41}$ so that $\Delta\phi = -0.948825313 \times 10^{-2}$ arcsec. This value of $\Delta\phi$ expresses the angle of surplus rather than angle of deficit (Dyer & Marleau 1995; Rahaman et al. 2005a). We plot U vs. ϕ (see Fig. 6).

Case II: $D = \text{odd} = 2n - 1$

$$U = \frac{\cos\phi}{R_0} + \frac{\mu n}{2^{2n-2}R_0^{2n-1}} \left[\frac{\cos(2n-1)\phi}{-(2n-1)^2 + 1} \right. \\ \left. + \frac{^{2n-1}C_1 \cos(2n-3)\phi}{-(2n-3)^2 + 1} + \dots \right] + \mu n \frac{^{2n-1}C_{n-1}}{2^{2n-2}R_0^{2n-1}} \frac{\phi \sin\phi}{2} \quad (28)$$

(i): $n = 2$

In this subcase

$$U = \frac{\cos\phi}{R_0} + \frac{2\mu}{R_0^3} \left[\frac{3\phi}{8}\sin\phi - \frac{1}{32}\cos 3\phi \right]. \quad (29)$$

For $U = 0$, we get $\cos\phi = -0.2255226204 \times 10^{-41}$. We plot U vs. ϕ (see Fig. 7). Here the net deflection of the ray is given by

$$\Delta\phi = -0.948825313 \times 10^{-2} \text{ arcsec}, \quad (30)$$

which is nothing but angle of surplus.

(ii): $n = 3$

In this subcase

$$U = \frac{\cos\phi}{R_0} + \frac{3\mu}{R_0^5} \left[\frac{5\phi}{16}\sin\phi - \frac{5}{128}\cos 3\phi - \frac{1}{384}\cos 5\phi \right]. \quad (31)$$

For $U = 0$, we get $\cos\phi = -0.2255226204 \times 10^{-41}$. We plot U vs. ϕ (see Fig. 8). Here the net deflection of the ray is given by

$$\Delta\phi = -0.948825313 \times 10^{-2} \text{ arcsec}, \quad (32)$$

which is again angle of surplus.

Table 3. Deflection of Starlight for HD-Models

Dimensions (N)	Deflection ($\Delta\phi$) (arcsec)
4	1.741300716
5	$-0.948825313 \times 10^{-2}$
6	...
7	$-0.948825313 \times 10^{-2}$
8	$-0.948825313 \times 10^{-2}$

Thus, we observe that when $D = 2$, viz., the total dimensions, $N = D + 2 = 4$ then only the result does agree with the observational data and $D > 2$, i.e., $N > 4$ is not compatible with solar system (see Table 3). It can be noted, from the Figs. 4 - 8, that all the trajectories of the light rays almost same for all D due to the factor $\cos\phi/R_0$ which is the dominating one. The trajectory of light will show different graph for large D .

Historically, it is important to note that on the basis of his ‘corpuscular’ theory including laws of mechanics and gravitation, Newton (1704) raised the pertinent issue that “Do not Bodies act upon Light at a distance, and by their action bend its Rays, and is not this action strongest at the least distance?” He calculated the amount of bending of light rays for Sun as $2m/r_0$. For $m = 1475$ meters, in the gravitational units, and $r_0 = 6.95 \times 10^8$ meters this equals 0.875 arcsec. However, though prediction of bending by Einstein (1911) was at first identical to that of Newton but later on he (1915) got the angular deflection of light as *twice* the size he predicted earlier which caused due to the general relativistic effect of the curved space-time. In 1919 scientific expeditions performed at Sobral in South America and Principe in West Africa by the leadership of Eddington. The reported observational results of angular deflections due to the solar eclipse were 1.98 ± 0.16 and 1.61 ± 0.40 arcsec, respectively. The mean of these two data was taken as confirmation of Einstein’s prediction of 1.75 arcsec (see Table 2). However, the experiments of Eddington and his co-workers had only 30 percent accuracy where the results were scattered between one half and twice the Einstein value (Will 2001). An analysis of large amount of Very Long Baseline Interferometry (VLBI) observations has shown that the ratio of the actual observed deflections to the deflections predicted by general relativity is very close to unity (e.g., 0.9996 ± 0.0017 (Lebach et al. 1995), 0.99994 ± 0.00031 (Eubanks et al. 1999), 0.99992 ± 0.00023 (Shapiro, Davis, Lebach & Gregory 2004)).

3.3 Gravitational Redshift

GR predicted that the frequency of the light would be affected due to gravitational field and is observable as a shift of spectral lines towards the red end of the spectrum. Pound-Rebka-Snider (Pound & Rebka 1959, 1960; Pound & Snider 1964) confirmed this effect through their precision test, sometimes known as Harvard Tower Experiment. In their first test they measured the redshift experienced by a 14.4 Kev γ -rays from the decay of Fe^{57} for a height of 22.5 meter tower and found $z = 2.57 \pm 0.26 \times 10^{-15}$.

Now, as usual, gravitational redshift for the solar system can be defined as

$$z = \frac{\Delta\gamma}{\gamma} = \left[\frac{g_{tt}(R^*)}{g_{tt}(R)} \right]^{1/2} - 1 \quad (33)$$

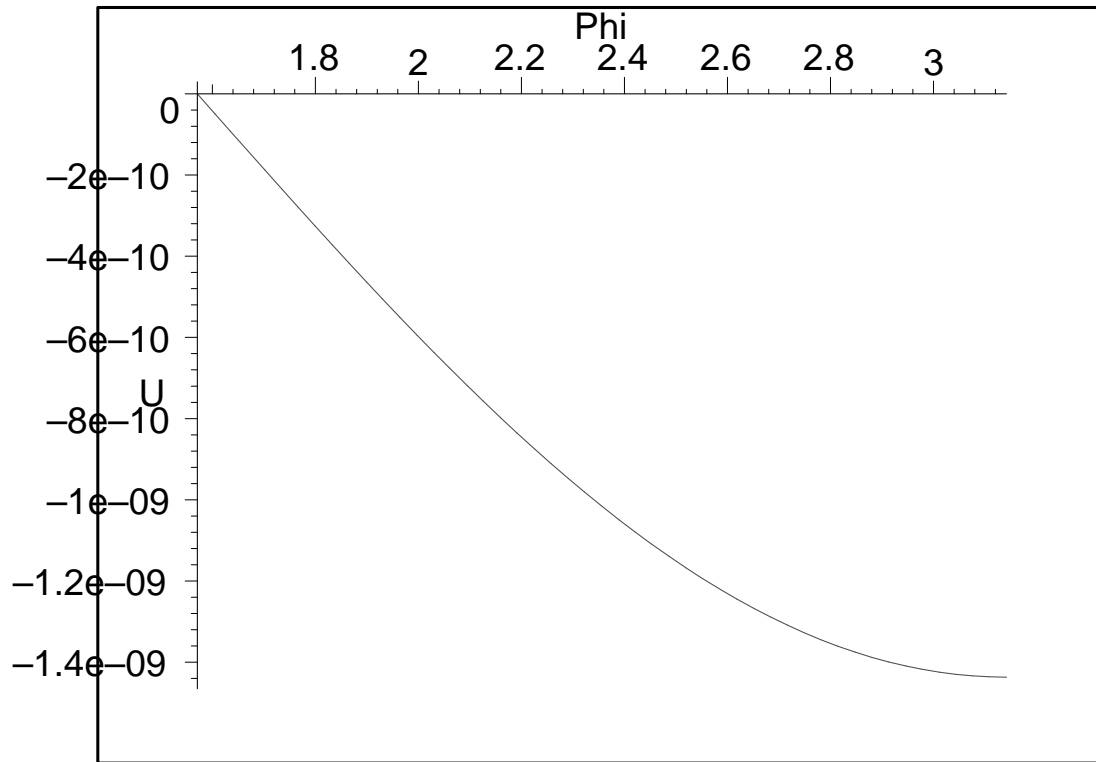


Figure 5. The plot U vs. ϕ for $D = 4$.

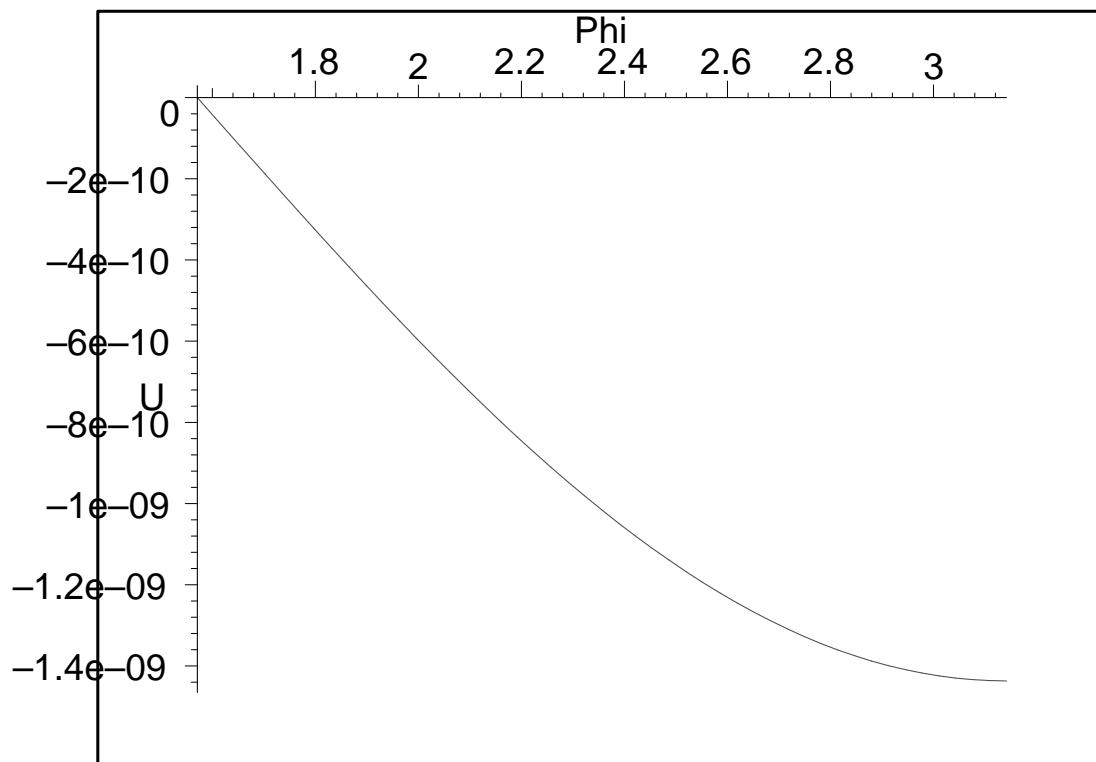


Figure 6. The plot U vs. ϕ for $D = 6$.

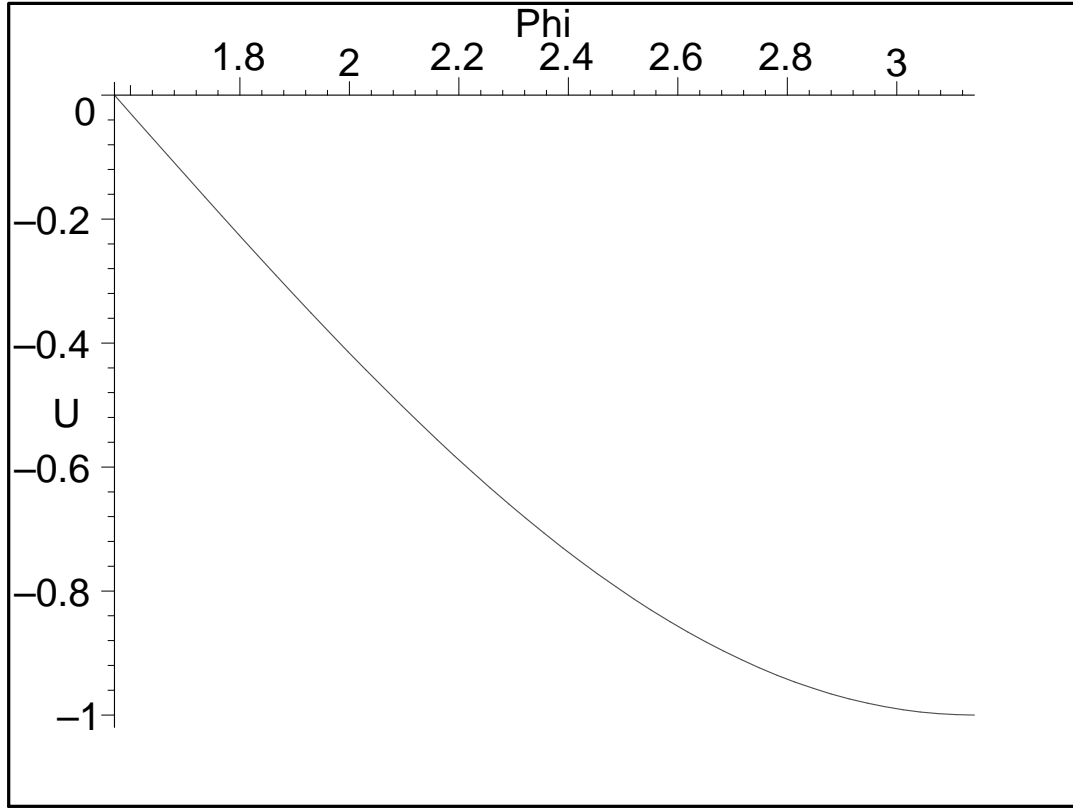


Figure 7. The plot U vs. ϕ for $D = 3$.

where R is the radius of the sun and R^* is the radius of the earth's orbit around sun.

Therefore, in view of the given HD line element (1), the metric tensors involved in the above equation (33) reduce to

$$\left[\frac{g_{tt}(R^*)}{g_{tt}(R)} \right]^{1/2} = \left[\frac{1 - \frac{\mu}{R^{D-1}}}{1 - \frac{\mu}{R^{D-1}}} \right]^{1/2} \cong \left[1 + \frac{\mu}{2R^{D-1}} - \frac{\mu}{2R^{D-1}} \right]. \quad (34)$$

By substituting the expressions of equation (34) in equation (33) for the assumption $R^* \gg \mu$, we get

$$z = \frac{\Delta\gamma}{\gamma} = \frac{\mu}{2R^{D-1}}. \quad (35)$$

Thus, in the Sun-Earth system we observe that for the usual 4-dimensional case ($D = 2$), gravitational redshift becomes $z \sim 2.12 \times 10^{-6} = z_2$ (say). Therefore, for $D > 2$, $z < z_2$ which indicates that as dimension increases the redshift gradually decreases (see Fig. 9). It can be also observed that redshift gradually increases with mass (since $z \propto \mu = 16\pi GM/Dc^2\Omega_D$, when radial distance and dimensions remain fixed in equation (35)). Thus, it seems that dimension acts as inversely proportional to mass of the gravitating body.

3.4 Gravitational Time Delay

Gravitational time delay, also known as Shapiro time delay which was reported by Shapiro (1964) is basically the effect of radar signals passing near a massive object take slightly longer time for a

round trip as measured by the observer than it would be in the absence of the object there. To proceed on towards the 'Fourth Test of General Relativity' let us consider the equation (12) in the form

$$t(r, r_0) = \int_{r_0}^r \frac{dr}{\left(1 - \frac{\mu}{r^{D-1}}\right) \left[1 - \frac{1 - \frac{\mu}{r_0^{D-1}}}{1 - \frac{\mu}{r_0^{D-1}}} \left(\frac{r_0}{r}\right)^2\right]^{1/2}} \quad (36)$$

which, after simplification, yields

$$t(r, r_0) = \int_{r_0}^r \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \times \left[1 + \frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2)r^{D-1}r_0^{D-3}}\right] dr. \quad (37)$$

Hence, transit time of the light ray from Mercury to Earth can be given by

$$t = \int_{r_0}^{r_1} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[1 + \frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2)r^{D-1}r_0^{D-3}}\right] dr + \int_{r_0}^{r_2} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[1 + \frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2)r^{D-1}r_0^{D-3}}\right] dr. \quad (38)$$

In the absence of the gravitational field (viz., $\mu = 0$) one can get

$$t_0 = \int_{r_0}^{r_1} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} dr + \int_{r_0}^{r_2} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} dr. \quad (39)$$

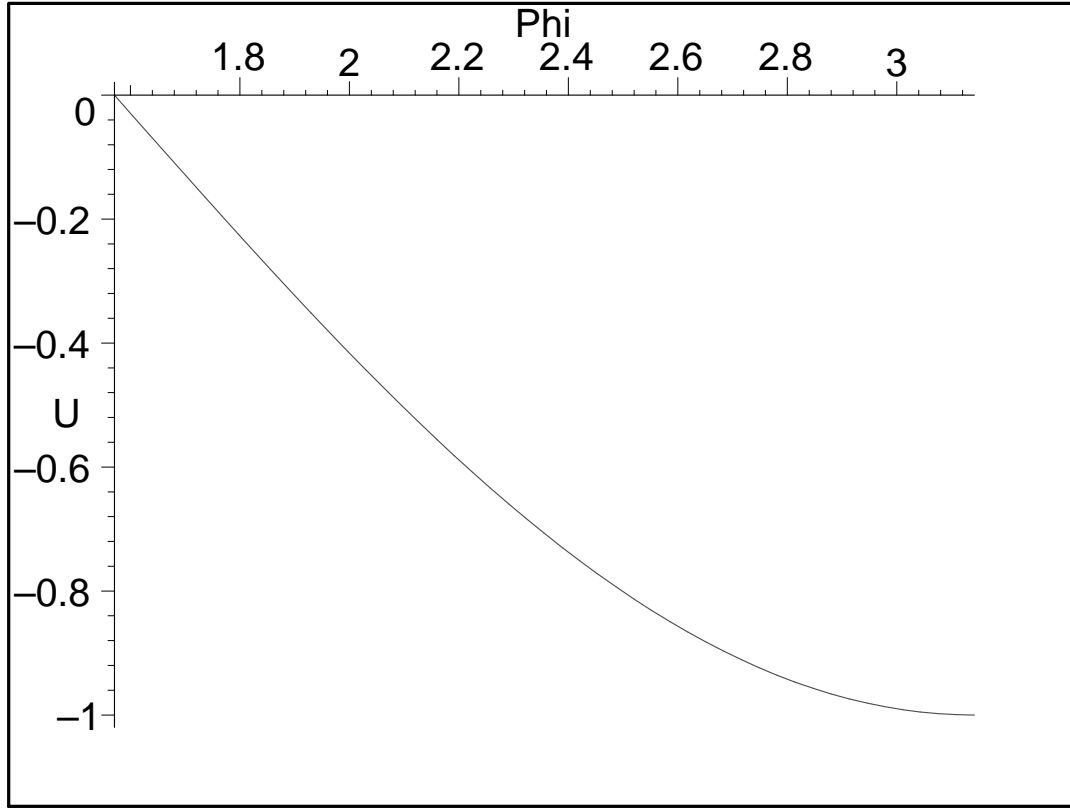


Figure 8. The plot U vs. ϕ for $D = 5$.

Hence, time delay for a round trip is

$$\begin{aligned} \Delta t &= 2(t - t_0) \\ &= 2 \int_{r_0}^{r_1} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[\frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2)r^{D-1}r_0^{D-3}} \right] dr \\ &+ 2 \int_{r_0}^{r_2} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[\frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2)r^{D-1}r_0^{D-3}} \right] dr. \quad (40) \end{aligned}$$

Let us consider

$$I = \int_{r_0}^{r_1} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[\frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2)r^{D-1}r_0^{D-3}} \right] dr. \quad (41)$$

The solution to this equation (41) is then given by for the following cases:

(i): $D = 2$

In this case the integral in the equation (41) becomes

$$I = \mu \ln \left[r + \sqrt{r^2 - r_0^2} \right]_{r_0}^r + \frac{\mu}{2} \left[\sqrt{\frac{r - r_0}{r + r_0}} \right]_{r_0}^r \quad (42)$$

so that the time delay for a round trip can be given as

$$\begin{aligned} \Delta t &= \frac{4GM}{c^2} \ln \left[\frac{(r_1 + \sqrt{r_1^2 - r_0^2})(r_2 + \sqrt{r_2^2 - r_0^2})}{r_0^2} \right] + \\ &\quad \frac{2GM}{c^2} \left[\sqrt{\frac{r_1 - r_0}{r_1 + r_0}} + \sqrt{\frac{r_2 - r_0}{r_2 + r_0}} \right]. \quad (43) \end{aligned}$$

If, however, $r_0 \ll r_1$ and $r_0 \ll r_2$, then

$$\Delta t = \frac{4GM}{c^2} \left[1 + \ln \frac{4r_1 r_2}{r_0^2} \right]. \quad (44)$$

The above expression for radar echo delay is in accordance with the standard literature (Weinberg 2004) when the Schwarzschild space-time is of usual four-dimensional entity and provides an amount 240 μsec as the maximum excess time delay for the Earth-Mercury system.

(ii): $D = 3$

Here

$$I = \frac{3\mu}{2r_0} \sec^{-1} \left(\frac{r}{r_0} \right). \quad (45)$$

Hence, the time delay in this case becomes

$$\Delta t = \frac{3\mu}{r_0} \left[\sec^{-1} \left(\frac{r_1}{r_0} \right) + \sec^{-1} \left(\frac{r_2}{r_0} \right) \right]. \quad (46)$$

Let us consider that either $x = (r_1/r_0) \gg 1$ or $x = (r_2/r_0) \gg 1$ so that, after neglecting the higher order terms like $1/x^3$, $1/x^5$, ... etc. we get

$$\sec^{-1} x \cong \frac{\pi}{2} - \frac{1}{x} \quad (47)$$

so that

$$\Delta t \cong \frac{4GM}{c^2} \left[\frac{1}{r_0} - \frac{1}{\pi} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \right]. \quad (48)$$

(iii): $D = 4$

For this case we have

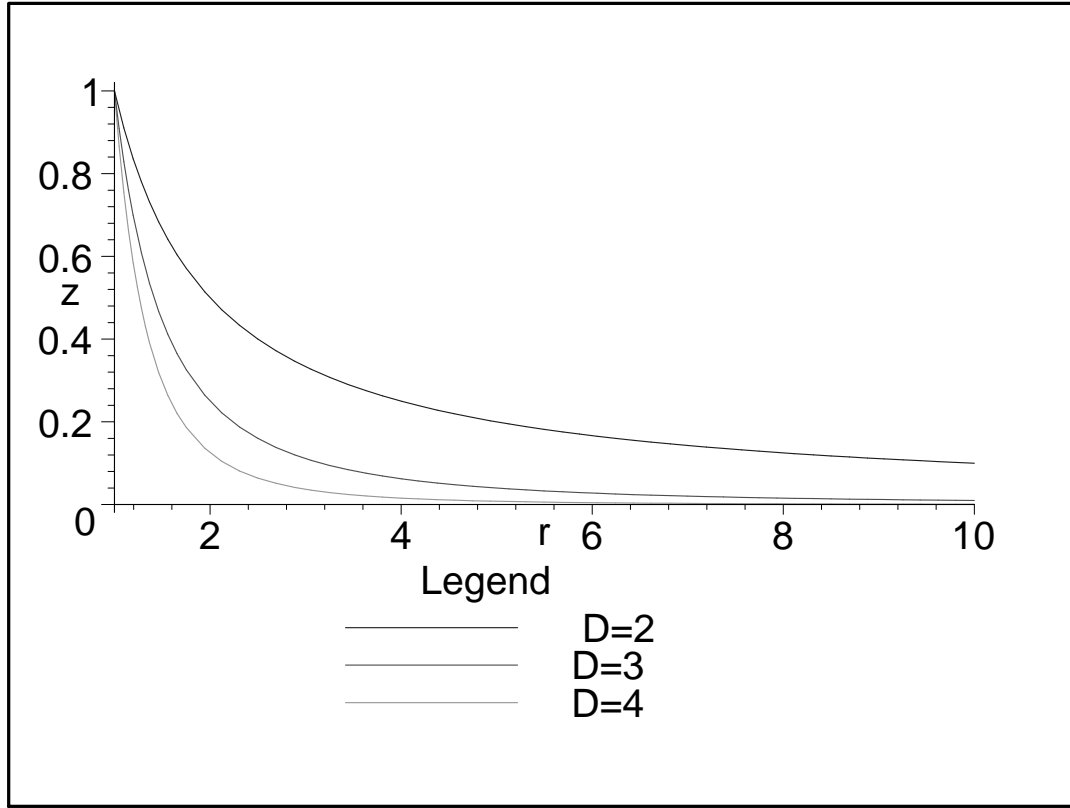


Figure 9. The plot Redshift (z) vs. r for different dimensions.

$$I = \frac{3GM}{2\pi c^2} \left[\frac{\sqrt{r^2 - r_0^2}}{rr_0^2} + \frac{1}{r_0^2} \sqrt{\frac{r - r_0}{r + r_0}} \right]_{r_0}^r. \quad (49)$$

Therefore, the expression for time delay becomes

$$\Delta t = \frac{3GM}{\pi c^2 r_0^2} \left[\frac{\sqrt{r_1^2 - r_0^2}}{r_1} + \frac{\sqrt{r_2^2 - r_0^2}}{r_2} + \sqrt{\frac{r_1 - r_0}{r_1 + r_0}} + \sqrt{\frac{r_2 - r_0}{r_2 + r_0}} \right]. \quad (50)$$

Thus, from the above case studies one can observe that the maximum time delay will occur when $D = 2$, i.e., for the usual 4-dimensional Schwarzschild space-time. Time delay decreases due to increase dimensions. This can be shown easily by assuming $r_0 \ll r_1$ and $r_0 \ll r_2$.

3.5 Motion of Test Particle

Let us consider a test particle having mass m which is moving in the gravitational field of a $D + 2$ -dimensional spacetime described by the metric (1). So, the Hamilton-Jacobi (HJ) equation for the test particle is (Chakraborty 1996; Chakraborty & Biswas 1996; Rahaman et al. 2005b)

$$g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} + m^2 = 0 \quad (51)$$

where g_{ik} are the classical background field and S is the Hamilton's characteristic function. For the metric (1) the explicit form of HJ

equation (51) now takes the form as

$$-\frac{1}{f} \left(\frac{\partial S}{\partial t} \right)^2 + f \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left[\left(\frac{\partial S}{\partial x_1} \right)^2 + \left(\frac{\partial S}{\partial x_2} \right)^2 + \dots + \left(\frac{\partial S}{\partial x_{D-2}} \right)^2 \right] + m^2 = 0 \quad (52)$$

where x_1, x_2, \dots, x_{D-2} are the independent coordinates on the surface of the unit $(D - 2)$ sphere such that

$$\begin{aligned} d\Omega_{D-2}^2 &= dx_1^2 + dx_2^2 + \dots + dx_{D-2}^2 \\ &\equiv d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \dots + \sin^2 \theta_1 \sin^2 \theta_2 \dots \sin^2 \theta_{D-3} d\theta_{D-2}^2 \end{aligned} \quad (53)$$

and f is given by $f(r) = 1 - \mu/r^{D-1}$ as introduced earlier.

In order to solve the above partial differential equation (52), let us choose the HJ function S as

$$S = -Et + S_1(r) + p_1.x_1 + p_2.x_2 + \dots + p_{D-2}.x_{D-2} \quad (54)$$

where E is identified as the energy of the particle and p_1, p_2, \dots, p_{D-2} are the momenta of the particle along different axes on the $(D - 2)$ sphere with the resulting momentum of the particle, $p = \sqrt{p_1^2 + p_2^2 + \dots + p_{D-2}^2}$.

Now, substitution of the *ansatz* (54) in equation (52) provides the following expression for the unknown function S_1 which is

$$S_1(r) = \epsilon \int \sqrt{\frac{E^2}{f^2} - \frac{m^2}{f} - \frac{p^2}{r^2 f}} dr \quad (55)$$

with $\epsilon = \pm 1$, where the sign changes whenever r passes through a zero of the integral (55).

To determine the trajectory of the particle following HJ method, let us consider that $\frac{\partial S}{\partial E} = \text{constant}$ and $\frac{\partial S}{\partial p_i} = \text{constant}$ [$i = 1, 2, \dots, (D-2)$]. Here we have chosen the constants to be zero without any loss of generality.

Therefore, based on the above assumptions one obtains the following two integrals

$$t = \epsilon \int \frac{\frac{E}{f^2}}{\sqrt{\frac{E^2}{f^2} - \frac{m^2}{f} - \frac{p^2}{r^2 f}}} dr, \quad (56)$$

$$x_i = \epsilon \int \frac{\left(\frac{p_i}{r^2 f}\right)}{\sqrt{\frac{E^2}{f^2} - \frac{m^2}{f} - \frac{p^2}{r^2 f}}} dr. \quad (57)$$

The radial velocity of the particle is then given, from equation (56), by

$$\frac{dr}{dt} = \frac{\sqrt{\frac{E^2}{f^2} - \frac{m^2}{f} - \frac{p^2}{r^2 f}}}{\frac{E}{f^2}}. \quad (58)$$

The turning points of the trajectory can be characterized by $\frac{dr}{dt} = 0$ and as a consequence the potential curve becomes

$$\frac{E}{m} = \sqrt{f} \left[1 + \frac{p^2}{m^2 r^2} \right]^{1/2} \equiv V(r) \quad (59)$$

so that one can write the effective potential, $V(r)$, in the form

$$V^2 = \left(1 - \frac{\mu}{r^{D-1}} \right) \left(1 + \frac{p^2}{m^2 r^2} \right) \quad (60)$$

Now, in a stationary system of energy E , the effective potential V must have an extremal value. Therefore, the condition to be imposed on the value of r for which energy attains its extremal one can be given by $\frac{dV}{dr} = 0$ so that

$$2p^2 r^{D-1} - \mu(D-1)m^2 r^2 - p^2 \mu(D+1) = 0. \quad (61)$$

It has at least one positive root the last term being negative (for $D > 3$). Thus, particles can be trapped by gravitational field of higher dimensional Schwarzschild space-time and hence the gravitational field is attractive in nature. For $D = 2$ and $D = 3$, we have some restrictions to get bound orbit as $p^2 > 3\mu^2 m^2$ and $p^2 > \mu m^2$ respectively. The plot V^2 vs. r for $D = 2$ and $D = 3$ have been provided in Figs. 10 and 11 respectively.

4 CONCLUSIONS

Our analytically performed solar system tests for GR with HD can be summarized as follows -

1. Perihelion shift: In $4D$ our result exactly coincides with that of Einstein's predicted value with an elliptical path followed by the planet Mercury. As we go increase on dimensions the paths rapidly become irregular in shapes and hence HD do not work at all.

2. Bending light: Here also our theoretical result is in good agreement with the experimental result 1.741300716 arcsec which become enormously different with an angle of surplus value -0.00948825313 arcsec $D > 2$.

3. Gravitational redshift: We observe that in the 4-dimensional case gravitational redshift becomes $z \sim 2.12 \times 10^{-6}$ in the Sun-Earth system. However, for $D > 2$ redshift gradually decreases

with the increase of dimensions such that dimension acts as inversely proportional to mass of the gravitating body. It can also be observed that for constant radial distant and dimension the redshift gradually increases with the mass of the planets.

4. Gravitational time delay: It is seen from the present investigation that radar echo delay is as usual in the case of $4D$ and decreases with increase of dimensions.

5. Motion of a test particle: Here the observation is that particles can be trapped by gravitational field of higher dimensions and hence the gravitational field is attractive in nature (with the restrictions to get bound orbit as $p^2 > 3\mu^2 m^2$ and $p^2 > \mu m^2$ for $D = 2$ and $D = 3$ respectively). Therefore, this is the only case under our study which is fairly compatible with the HD version of GR.

In a nutshell, our overall observation regarding HD realm of GR is, in general, similar to that of Liu & Overduin (2000) which is as follows: "... the existence of small but potentially measurable departures from the standard $4D$ Einstein predictions". However, in some of our HD cases invoke the word 'drastic' in place of 'small' one!

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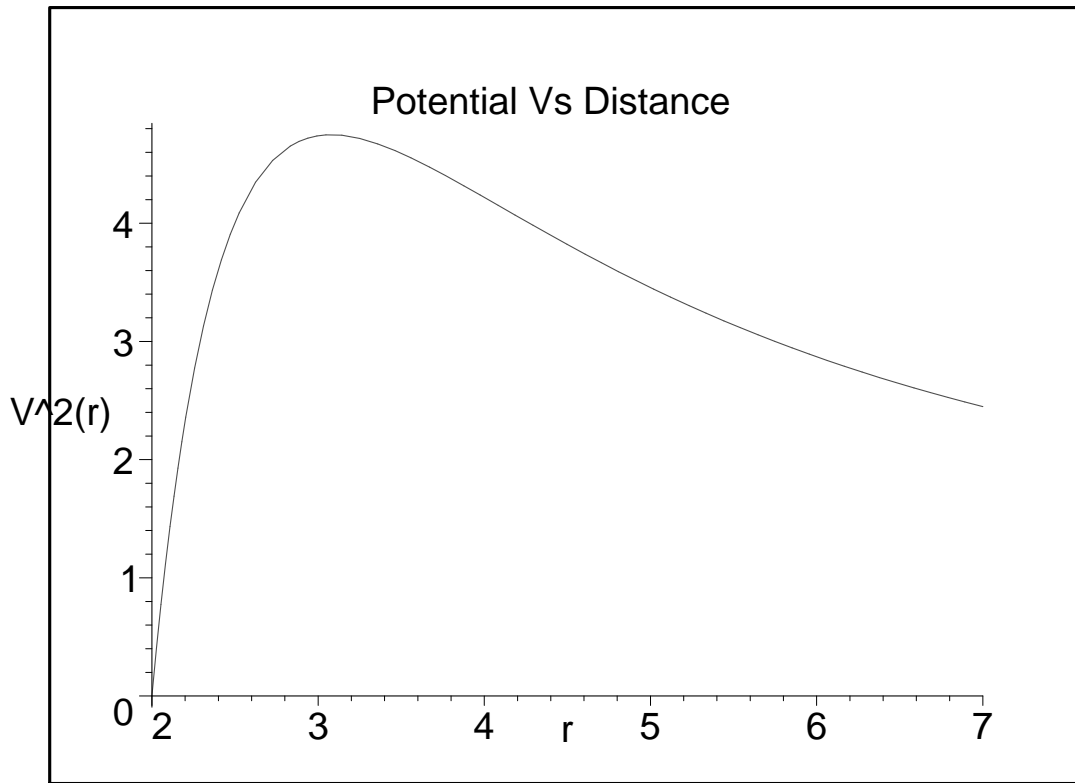


Figure 10. The plot V^2 vs. r for $D = 2$.

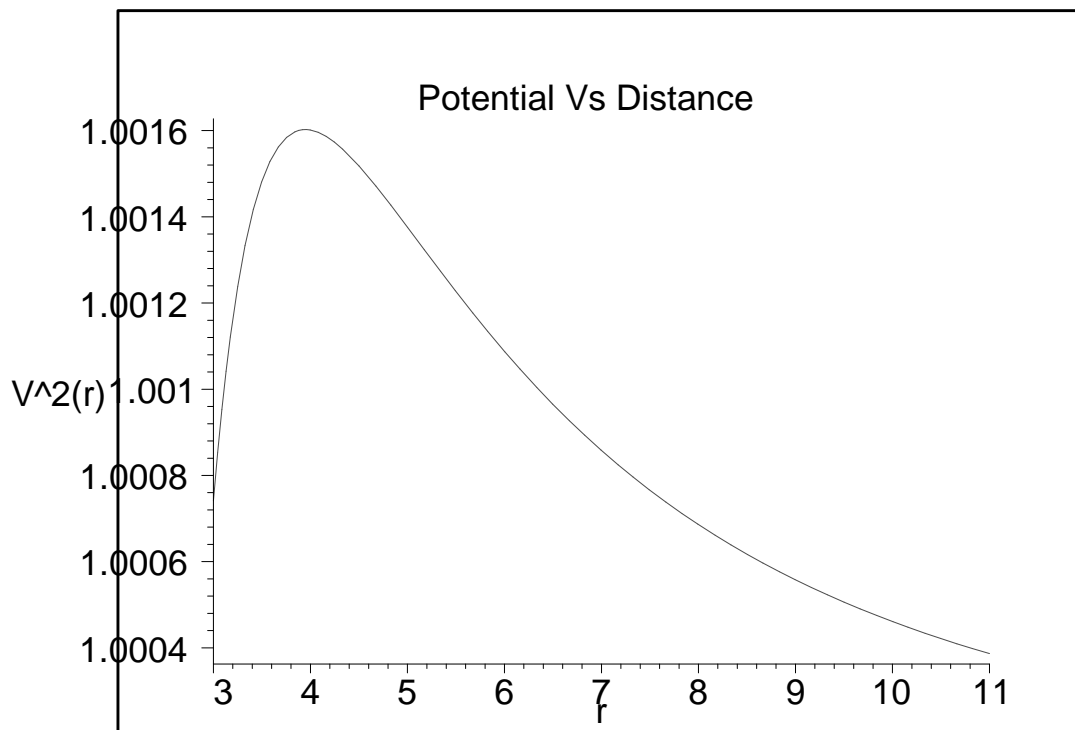


Figure 11. The plot V^2 vs. r for $D = 3$.

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